Powers of matrices, polynomials in matrices

Let, A be an n square matrix over a field K. Powers of A are defined as follows:

$$A^2 = AA, \quad A^3 = A^2A, \; \ldots, \; A^{n+1} = A^nA \ldots$$

$$A^0 = I$$

Polynomials in the matrix A are also defined. Specifically, for any polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where, the a_i are scalars in K; $f(A)$ is defined to be the following matrix

$$f(x) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$$

Let, $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$$

and $A^3 = A^2A = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

$$= \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$$

Suppose, $f(x) = 2x^2 - 3x + 5$

$g(x) = x^2 + 3x - 10$

$$f(A) = 2\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}$$

$$g(A) = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Invertible matrices or Nonsingular matrices:-

A square matrix A is said to be invertible or nonsingur if there exist a matrix B such that:

$$AB = BA = I$$

where, I is the identity matrix. such a matrix B is unique. that is, if $AB_1 = B_1A = I$ & $AB_2 = B_2A = I$ then;

$$B_1 = B_1I = B_1(AB_2) = (B_1A)B_2 = IB_2 = B_2$$

A matrix B the inverse of A and denote it by A^{-1}

Example: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ A & B are inverse.

Inverse of a 2x2 matrix

Let, A be an arbitrary 2x2 matrix

say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = ?$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $ax_1 + by_1 = 1$ $\quad ax_2 + by_2 = 0$

$cx_1 + dy_1 = 0$ $\quad cx_2 + dy_2 = 1$

$|A| = ad - bc$ (the determinant of A)

$|A| \neq 0$

$$x_1 = \frac{d}{|A|}, \quad y_1 = \frac{-c}{|A|}, \quad x_2 = \frac{-b}{|A|}, \quad y_2 = \frac{a}{|A|}$$

$$A^{-1} = \begin{bmatrix} a & d \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$|A| \neq 0$ & i) Inter change of two elements on the diagonal

ii) Take the negatives of the other two elements

iii) multiply the resulting matrix by $\frac{1}{|A|}$.

If $|A| = 0$, the matrix A is not invertible.

<u>Example:</u>

Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$|A| = 2 \times 5 - 3 \times 4 = 10 - 12 = -2 \quad \therefore |A| \neq 0$

$$\therefore A^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

& $|B| = 1 \times 6 - 3 \times 2 = 6 - 6 = 0 \quad \therefore |A| = 0$

$\therefore$ B has no inverse

Diagonal and Triangular matrices

A square matrix $D = [d_{ij}]$ is diagonal if its nondiagonal entries are all zero.

$$D = \text{diag}(d_{11}, d_{22}, \ldots, d_{nn})$$

Let,
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 6 & & & \\ & 0 & & \\ & & -9 & \\ & & & 8 \end{bmatrix}$$

A square matrix $A = [a_{ij}]$ is upper triangular or simply triangular matrix.

Let,
$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ & b_{22} & b_{23} \\ & & b_{33} \end{bmatrix}$$

& lower triangular

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & & \\ b_{21} & b_{22} & \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Special Real Square matrices

a) Symmetric Matrices:-

A matrix A is symmetric if $A^T = A$

A matrix A is skew-symmetric if $A^T = -A$

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Example:- Let, $A = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\because A^T = A$ $\therefore$ A is symmetric

$\because$ diagonal elements of B are zero (0) and symmetric elements are negatives of each other or $B^T = -B$

$\therefore$ B is skew symmetric

Since C is not square, $\therefore$

$\therefore$ ~~C is neigther~~

$\therefore$ C is neither symmetric nor skew symmetric.

(b) orthogonal Matrices

A real matrix A is orthogonal if $A^T = A^{-1}$

$AA^T = A^TA = I$

& A must necessarily be square and invertible

Examples: Let, $A = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{4}{9} & -\frac{4}{9} & -\frac{7}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$

if $AA^T = I$ $\therefore$ $A^T = A^{-1}$ i.e A is orthogonal.

Normal vectors:

A real matrix A is normal if it commutes with its transpose A^T.

i,e

$$AA^T = A^TA$$

again, if A is symmetric, orthogonal or skew symmetric then A is normal.

Let, $A = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}$$

$\therefore AA^T = A^TA$

$\therefore A$ is normal.